# A REVERSE OF THE CAUCHY-BUNYAKOVSKY-SCHWARZ INTEGRAL INEQUALITY FOR COMPLEX-VALUED FUNCTIONS AND APPLICATIONS FOR FOURIER TRANSFORM

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Abstract. A reverse of the Cauchy-Bunyakovsky-Schwarz integral inequality for complex-valued functions and applications for the finite Fourier transform are given.

## 1. Introduction

Let $(\Omega, \Sigma, \mu)$ be a measure space consisting of a set $\Omega$, $\Sigma$ a $\sigma-$algebra of parts and $\mu$ a countably additive and positive measure on $\Sigma$ with values in $\mathbb{R} \cup \{\infty\}$. Let $\rho \geq 0$ be a $\mu-$measurable function on $\Omega$. Denote by $L^2_\rho(\Omega, \mathbb{K})$ the Hilbert space of all real or complex valued functions defined on $\Omega$ and $2-\rho-$integrable on $\Omega$, i.e.,

$$\int_\Omega \rho(s) |f(s)|^2 d\mu(s) < \infty. \tag{1.1}$$

If $f, g : \Omega \to \mathbb{R}$ are real functions such that there exist the constants $0 < m \leq M < \infty$ with the property that

$$m \leq \frac{f(s)}{g(s)} \leq M \quad \text{for} \quad \mu-\text{a.e. } s \in \Omega, \tag{1.2}$$

then we have

$$\frac{\int_\Omega \rho(s) f^2(s) d\mu(s) \int_\Omega \rho(s) g^2(s) d\mu(s)}{\left(\int_\Omega \rho(s) f(s) g(s) d\mu(s)\right)^2} \leq \frac{(M+m)^2}{4mM}. \tag{1.3}$$

This inequality (in its discrete version) is known in the literature as the Cassels inequality (see for instance [8]).

If we assume that there exists the constants $m_i, M_i$ $(i = 1, 2)$ such that

$$\begin{aligned} 0 < m_1 \leq f(s) \leq M_1 < \infty \text{ for } \mu-\text{a.e. } s \in \Omega, \\ 0 < m_2 \leq g(s) \leq M_2 < \infty \text{ for } \mu-\text{a.e. } s \in \Omega, \end{aligned} \tag{1.4}$$

then from Cassels' inequality, we deduce the Pólya-Szegö weighted inequality, [7], which is also known in the literature as the Greub-Reinboldt's inequality [6]:

$$\frac{\int_\Omega \rho(s) f^2(s) d\mu(s) \int_\Omega \rho(s) g^2(s) d\mu(s)}{\left(\int_\Omega \rho(s) f(s) g(s) d\mu(s)\right)^2} \leq \frac{(M_1 M_2 + m_1 m_2)^2}{4 m_1 m_2 M_1 M_2}. \tag{1.5}$$

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In the recent paper [4], S.S. Dragomir obtained the following extension for real or complex-valued functions of the Cassels' inequality (see Proposition 4, [4]):

Let $f,g\in L_{\rho}^{2}(\Omega,\mathbb{K})$ and $\gamma,\Gamma\in\mathbb{K}$ such that $\operatorname{Re}(\Gamma\bar{\gamma})\geq 0$. If either

$$\operatorname{Re}\left[(\Gamma g(s)-f(s))\left(\overline{f(s)}-\bar{\gamma}\overline{g(s)}\right)\right]\geq 0 \text{ for } \mu-\text{a.e. } s\in\Omega \tag{1.6}$$

or, equivalently,

$$\left|f(s)-\frac{\Gamma+\gamma}{2}\cdot g(s)\right|\leq\frac{1}{2}|\Gamma-\gamma|\,|g(s)| \text{ for } \mu-\text{a.e. } s\in\Omega, \tag{1.7}$$

then we have the inequality

$$\begin{aligned}
&\int_{\Omega}\rho(s)|f(s)|^{2}d\mu(s)\int_{\Omega}\rho(s)|g(s)|^{2}d\mu(s) \\
&\leq\frac{1}{4}\cdot\frac{\left\{\operatorname{Re}\left[\left(\bar{\Gamma}+\bar{\gamma}\right)\right]\int_{\Omega}\rho(s)f(s)\overline{g(s)}d\mu(s)\right\}^{2}}{\operatorname{Re}(\Gamma\bar{\gamma})} \\
&\leq\frac{1}{4}\cdot\frac{|\Gamma+\gamma|^{2}}{\operatorname{Re}(\Gamma\bar{\gamma})}\left|\int_{\Omega}\rho(s)f(s)\overline{g(s)}d\mu(s)\right|^{2}.
\end{aligned} \tag{1.8}$$

The constant $\frac{1}{4}$ is best possible in both inequalities.

If (1.6) or (1.7) holds true, then the following additive version of (1.8) also holds [4]

$$\begin{aligned}
0&\leq\int_{\Omega}\rho(s)|f(s)|^{2}d\mu(s)\int_{\Omega}\rho(s)|g(s)|^{2}d\mu(s)-\left|\int_{\Omega}\rho(s)f(s)\overline{g(s)}d\mu(s)\right|^{2} \\
&\leq\frac{1}{4}\cdot\frac{|\Gamma-\gamma|^{2}}{\operatorname{Re}(\Gamma\bar{\gamma})}\left|\int_{\Omega}\rho(s)f(s)\overline{g(s)}d\mu(s)\right|^{2}.
\end{aligned} \tag{1.9}$$

Here $\frac{1}{4}$ is the best possible constant as well.

The main aim of this paper is to establish a reverse of the Cauchy-Bunyakovsky-Schwarz integral inequality for complex-valued functions that generalises earlier results of the first author and apply it for the approximation of the finite Fourier transform.

## 2. Some Reverses of the $(CBS)$ −Inequality

We start with the following lemma that is of interest in itself.

**Lemma 1.** *Let $f,g\in L_{\rho}^{2}(\Omega,\mathbb{K})$ with $g(s)\neq 0$ for $\mu-$a.e. $s\in\Omega$. If there exists the constants $\alpha\in\mathbb{K}$ and $r>0$ such that*

$$\frac{f(s)}{g(s)}\in\bar{D}(\alpha,r):=\{z\in\mathbb{K}|\,|z-\alpha|\leq r\}, \tag{2.1}$$

*for $\mu-$a.e. $s\in\Omega$, then we have the inequality*

$$\begin{aligned}
&\int_{\Omega}\rho(s)|f(s)|^{2}d\mu(s)+\left(|\alpha|^{2}-r^{2}\right)\int_{\Omega}\rho(s)|g(s)|^{2}d\mu(s) \\
&\leq 2\operatorname{Re}\left[\bar{\alpha}\int_{\Omega}\rho(s)f(s)\overline{g(s)}d\mu(s)\right] \\
&\leq 2|\alpha|\left|\int_{\Omega}\rho(s)f(s)\overline{g(s)}d\mu(s)\right|.
\end{aligned} \tag{2.2}$$

*The constant $c = 2$ in the right side of (2.2) is best possible in the sense that it cannot be replaced by a smaller constant.*

*Proof.* From (2.1) we have

$$|f(s) - \alpha g(s)|^2 \le r |g(s)|^2,$$

for $\mu-$a.e. $s \in \Omega$, which is clearly equivalent to

$$|f(s)|^2 + \left(|\alpha|^2 - r^2\right) |g(s)|^2 \le 2 \operatorname{Re}\left[\bar{\alpha}\left(f(s)\overline{g(s)}\right)\right], \tag{2.3}$$

for $\mu-$a.e. $s \in \Omega$.

Multiplying (2.3) with $\rho(s) \ge 0$ and integrating on $\Omega$, we deduce the first inequality in (2.2). The second inequality is obvious by the fact that $\operatorname{Re}(z) \le |z|$ for $z \in \mathbb{C}$ and we omit the details.

To prove the sharpness for the constant 2, assume that, under the hypothesis of the theorem, there exists a $c > 0$ such that

$$\begin{aligned} \int_\Omega \rho(s) |f(s)|^2 d\mu(s) + \left(|\alpha|^2 - r^2\right) \int_\Omega \rho(s) |g(s)|^2 d\mu(s) \\ \le c \operatorname{Re}\left[\bar{\alpha} \int_\Omega \rho(s) f(s) \overline{g(s)} d\mu(s)\right] \end{aligned} \tag{2.4}$$

provided $\frac{f(s)}{g(s)} \in \bar{D}(\alpha, r)$ for $\mu-$a.e. $s \in \Omega$.

If we choose $\rho$ such that $\int_\Omega \rho(s) d\mu(s) = 1$, $f(s) = 2r$, $g(s) = 1$ and $\alpha = r$, $r > 0$, then we have $\frac{f(s)}{g(s)} = 2r \in \bar{D}(r, r)$, and by (2.4) we deduce

$$4r^2 \le 2cr^2$$

giving $c \ge 2$. ∎

The case when the disk $\bar{D}(\alpha, r)$ does not contain the origin, i.e., $|\alpha| > r > 0$, provides the following interesting reverse of the Cauchy-Bunyakovsky-Schwarz inequality.

**Theorem 1.** *Let $f, g, \rho$ be as in Lemma 1 and assume that $|\alpha| > r > 0$. Then we have the inequality*

$$\begin{aligned} &\int_\Omega \rho(s) |f(s)|^2 d\mu(s) \int_\Omega \rho(s) |g(s)|^2 d\mu(s) \\ &\le \frac{1}{|\alpha|^2 - r^2} \left[\operatorname{Re}\left\{\bar{\alpha} \int_\Omega \rho(s) f(s) \overline{g(s)} d\mu(s)\right\}\right]^2 \\ &\le \frac{|\alpha|^2}{|\alpha|^2 - r^2} \left|\int_\Omega \rho(s) f(s) \overline{g(s)} d\mu(s)\right|^2. \end{aligned} \tag{2.5}$$

*The constant $c = 1$ in the first and second inequality is best possible in the sense that it cannot be replaced by a smaller constant.*

*Proof.* Since $|\alpha| > r$, we may divide (2.2) by $\sqrt{|\alpha|^2 - r^2} > 0$ to obtain

$$\int_\Omega \rho(s)|f(s)|^2 d\mu(s) + \sqrt{|\alpha|^2 - r^2}\int_\Omega \rho(s)|g(s)|^2 d\mu(s) \leq \frac{2}{\sqrt{|\alpha|^2 - r^2}} \operatorname{Re}\left[\bar{\alpha}\int_\Omega \rho(s) f(s)\overline{g(s)} d\mu(s)\right]. \tag{2.6}$$

On the other hand, by the use of the following elementary inequality

$$\frac{1}{\beta}p + \beta p \geq 2\sqrt{pq} \quad \text{for} \quad \beta > 0 \text{ and } p, q \geq 0, \tag{2.7}$$

we may state that

$$2\left(\int_\Omega \rho(s)|f(s)|^2 d\mu(s)\right)^{\frac{1}{2}}\left(\int_\Omega \rho(s)|g(s)|^2 d\mu(s)\right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{|\alpha|^2 - r^2}}\int_\Omega \rho(s)|f(s)|^2 d\mu(s) + \sqrt{|\alpha|^2 - r^2}\int_\Omega \rho(s)|g(s)|^2 d\mu(s). \tag{2.8}$$

Utilising (2.6) and (2.8), we deduce

$$\left(\int_\Omega \rho(s)|f(s)|^2 d\mu(s)\right)^{\frac{1}{2}}\left(\int_\Omega \rho(s)|g(s)|^2 d\mu(s)\right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{|\alpha|^2 - r^2}} \operatorname{Re}\left[\bar{\alpha}\int_\Omega \rho(s) f(s)\overline{g(s)} d\mu(s)\right],$$

which is clearly equivalent to the first inequality in (2.5). The second part of this inequality is obvious.

To prove the sharpness of the constant, assume that (2.5) holds with the quantity $c > 0$, i.e.,

$$\int_\Omega \rho(s)|f(s)|^2 d\mu(s)\int_\Omega \rho(s)|g(s)|^2 d\mu(s) \leq \frac{c}{|\alpha|^2 - r^2}\left[\operatorname{Re}\left\{\bar{\alpha}\int_\Omega \rho(s) f(s)\overline{g(s)} d\mu(s)\right\}\right]^2, \tag{2.9}$$

provided $\frac{f(s)}{g(s)} \in \bar{D}(\alpha, r)$ and $|\alpha| > r$.

Assume that $\mathbb{K} = \mathbb{R}$, $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 \neq \emptyset$, where $\Omega_1, \Omega_2$ are measurable sets, $\int_\Omega \rho(s) d\mu(s) = \frac{1}{2}$ and $f(s) = \alpha + r$, $s \in \Omega_1$, $f(s) = \alpha - r$, $s \in \Omega_2$, $g(s) = 1$, $s \in \Omega$, $\alpha > r$. Then $\frac{f(s)}{g(s)} \in \bar{D}(\alpha, r)$ for any $s \in \Omega$ and thus

$$\int_\Omega \rho(s)(f(s))^2 d\mu(s) = \int_{\Omega_1} \rho(s)(\alpha + r)^2 d\mu(s) + \int_{\Omega_2} \rho(s)(\alpha - r)^2 d\mu(s) = \frac{1}{2}\left[(\alpha + r)^2 + (\alpha - r)^2\right] = \alpha^2 + r^2,$$

$$\int_\Omega \rho(s) f(s) g(s) d\mu(s) = \int_{\Omega_1} \rho(s)(\alpha + r) d\mu(s) + \int_{\Omega_2} \rho(s)(\alpha - r) d\mu(s) = \frac{1}{2}[\alpha + r + \alpha - r] = \alpha.$$

Therefore, by (2.9), we deduce

$$\alpha^2 + r^2 \le \frac{c\alpha^4}{\alpha^2 - r^2} \quad \text{for} \quad \alpha > r > 0,$$

which is clearly equivalent to

$$(c-1)\,\alpha^4 + r^4 \ge 0 \quad \text{for any} \quad \alpha > r > 0.$$

If in this inequality we choose $\alpha = 1$, $r = q \in (0,1)$ and let $q \to 0+$, then we deduce $c \ge 1$. ∎

The following corollary is a natural consequence of the above theorem.

**Corollary 1.** *Under the assumptions of Theorem 1, we have the following additive reverse of the Cauchy-Bunyakovsky-Schwarz inequality:*

$$\begin{aligned} 0 &\le \int_\Omega \rho(s)\,|f(s)|^2\,d\mu(s) \int_\Omega \rho(s)\,|g(s)|^2\,d\mu(s) \\ &\quad - \left|\int_\Omega \rho(s)\,f(s)\,\overline{g(s)}d\mu(s)\right|^2 \\ &\le \frac{r^2}{|\alpha|^2 - r^2}\left|\int_\Omega \rho(s)\,f(s)\,\overline{g(s)}d\mu(s)\right|^2. \end{aligned} \tag{2.10}$$

*The constant $c = 1$ is best possible in the sense mentioned above.*

**Remark 1.** *If in Theorem 1, we assume that $|\alpha| = r$, then we obtain the inequality:*

$$\begin{aligned} \int_\Omega \rho(s)\,|f(s)|^2\,d\mu(s) &\le 2\,\mathrm{Re}\left[\bar{\alpha}\int_\Omega \rho(s)\,f(s)\,\overline{g(s)}d\mu(s)\right] \\ &\le 2\,|\alpha|\left|\int_\Omega \rho(s)\,f(s)\,\overline{g(s)}d\mu(s)\right|. \end{aligned} \tag{2.11}$$

*The constant 2 is sharp in both inequalities.*

*We also remark that if $r > |\alpha|$, then (2.2) may be written as*

$$\begin{aligned} &\int_\Omega \rho(s)\,|f(s)|^2\,d\mu(s) \\ &\le \left(r^2 - |\alpha|^2\right)\int_\Omega \rho(s)\,|g(s)|^2\,d\mu(s) + 2\,\mathrm{Re}\left[\bar{\alpha}\int_\Omega \rho(s)\,f(s)\,\overline{g(s)}d\mu(s)\right] \\ &\le \left(r^2 - |\alpha|^2\right)\int_\Omega \rho(s)\,|g(s)|^2\,d\mu(s) + 2\,|\alpha|\left|\int_\Omega \rho(s)\,f(s)\,\overline{g(s)}d\mu(s)\right|. \end{aligned} \tag{2.12}$$

The following particular case of interest also holds.

**Corollary 2.** *Let $f, g \in L^2_\rho(\Omega,\mathbb{K})$ with $g(s) \ne 0$ for $\mu-$a.e. $s \in \Omega$. If there exists the constants $\gamma, \Gamma \in \mathbb{K}$ with $\mathrm{Re}(\Gamma\bar{\gamma}) > 0$ and $\Gamma \ne \gamma$, so that either:*

$$\left|\frac{f(s)}{g(s)} - \frac{\gamma+\Gamma}{2}\right| \le \frac{1}{2}\,|\Gamma - \gamma| \quad \text{for } \mu - \text{a.e. } s \in \Omega, \tag{2.13}$$

*or, equivalently,*

$$\mathrm{Re}\left[\left(\Gamma - \frac{f(s)}{g(s)}\right)\left(\overline{\frac{f(s)}{g(s)}} - \gamma\right)\right] \ge 0 \quad \text{for } \mu - \text{a.e. } s \in \Omega, \tag{2.14}$$

*holds, then we have the inequalities*

$$\begin{aligned}(2.15)\qquad &\int_\Omega \rho(s)|f(s)|^2 d\mu(s)\int_\Omega \rho(s)|g(s)|^2 d\mu(s)\\ &\le \frac{1}{2\operatorname{Re}(\Gamma\bar{\gamma})}\left\{\operatorname{Re}\left[\left(\bar{\gamma}+\bar{\Gamma}\right)\int_\Omega \rho(s) f(s)\overline{g(s)}d\mu(s)\right]\right\}^2\\ &\le \frac{|\Gamma+\gamma|^2}{4\operatorname{Re}(\Gamma\bar{\gamma})}\left|\int_\Omega \rho(s) f(s)\overline{g(s)}d\mu(s)\right|^2.\end{aligned}$$

*The constants* $\frac{1}{2}$ *and* $\frac{1}{4}$ *are best possible in (2.15).*

*Proof.* The fact that the relations (2.13) and (2.14) are equivalent follows by the fact that for $z,u,U\in\mathbb{C}$, the following inequalities are equivalent

$$\left|z-\frac{u+U}{2}\right|\le\frac{1}{2}|U-u|$$

and

$$\operatorname{Re}\left[(U-z)(\bar{z}-\bar{u})\right]\ge 0.$$

Define $\alpha:=\frac{\gamma+\Gamma}{2}$ and $r=\frac{1}{2}|\Gamma-\gamma|$. Then

$$|\alpha|^2-r^2=\frac{|\Gamma+\gamma|^2}{4}-\frac{|\Gamma-\gamma|^2}{4}=\operatorname{Re}(\Gamma\bar{\gamma})>0.$$

Consequently, we may apply Theorem 1, and the inequalities (2.15) are proved.

The sharpness of the constants may be proven in a similar manner to that in the proof of Theorem 1, and we omit the details. ∎

**Remark 2.** *Note that the above result is due to S.S. Dragomir* [4] *and has been obtained in a different manner in the above mentioned reference.*

*If* $\gamma=m$, $\Gamma=M$ *and* $M>m>0$, *then from (2.15) we also recapture Cassels' result (1.3).*

The following additive version that can be easily derived from the above corollary is of interest (see also (1.9)).

**Corollary 3.** *With the assumptions of Corollary 2, we have the inequalities:*

$$\begin{aligned}(2.16)\qquad 0&\le\int_\Omega \rho(s)|f(s)|^2 d\mu(s)\int_\Omega \rho(s)|g(s)|^2 d\mu(s)\\ &\quad-\left|\int_\Omega \rho(s) f(s)\overline{g(s)}d\mu(s)\right|^2\\ &\le\frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})}\left|\int_\Omega \rho(s) f(s)\overline{g(s)}d\mu(s)\right|^2.\end{aligned}$$

*The constant* $\frac{1}{4}$ *is best possible in (2.16) in the sense that it cannot be replaced by a smaller quantity.*

## 3. A Pre-Grüss Type Inequality

The following result provides an inequality of pre-Grüss type that may be useful in applications when one of the factors is known and some bounds for the second factor are provided.

**Theorem 2.** *Let $\rho : \Omega \to [0,\infty)$ be a $\mu-$measurable function on $\Omega$ with the property that $\int_\Omega \rho(s)\,d\mu(s) = 1$. If $f, g \in L^2_\rho(\Omega,\mathbb{K})$ and there exist the constants $\varphi \in \mathbb{K}$ and $\delta > 0$ with $|\varphi| > \delta$ and such that $f(s) \in \bar{D}(\varphi,\delta)$ for $\mu-$a.e. $s \in \Omega$, then we have the inequality:*

$$\left|\int_\Omega \rho(s) f(s)\overline{g(s)}d\mu(s) - \int_\Omega \rho(s) f(s)\,d\mu(s)\int_\Omega \rho(s)\overline{g(s)}d\mu(s)\right| \le \left[\int_\Omega \rho(s)|g(s)|^2 d\mu(s) - \left|\int_\Omega \rho(s)\overline{g(s)}d\mu(s)\right|^2\right]^{\frac{1}{2}} \times \frac{\delta}{\sqrt{|\varphi|^2-\delta^2}}\left|\int_\Omega \rho(s) f(s)\,d\mu(s)\right|. \tag{3.1}$$

*The multiplicative constant $c = 1$ is best possible in (3.1).*

*Proof.* We know, by the following complex-valued Korkine's identity (which can be verified by simple computation with integrals), that

$$\int_\Omega \rho(s) f(s)\overline{g(s)}d\mu(s) - \int_\Omega \rho(s) f(s)\,d\mu(s)\int_\Omega \rho(s)\overline{g(s)}d\mu(s) = \frac{1}{2}\int_\Omega\int_\Omega \rho(s)\rho(t)[f(s)-f(t)]\left[\overline{g(s)}-\overline{g(t)}\right]d\mu(s)\,d\mu(t).$$

Applying Schwarz's inequality for double integrals we also have

$$\begin{aligned}
&\left|\int_\Omega\int_\Omega \rho(s)\rho(t)[f(s)-f(t)]\left[\overline{g(s)}-\overline{g(t)}\right]d\mu(s)\,d\mu(t)\right| \\
&\le \left(\int_\Omega\int_\Omega \rho(s)\rho(t)|f(s)-f(t)|^2 d\mu(s)\,d\mu(t)\right)^{\frac{1}{2}} \\
&\quad\times\left(\int_\Omega\int_\Omega \rho(s)\rho(t)|g(s)-g(t)|^2 d\mu(s)\,d\mu(t)\right)^{\frac{1}{2}} \\
&= \left(\int_\Omega \rho(s)|f(s)|^2 d\mu(s) - \left|\int_\Omega \rho(s) f(s)\,d\mu(s)\right|^2\right)^{\frac{1}{2}} \\
&\quad\times\left(\int_\Omega \rho(s)|g(s)|^2 d\mu(s) - \left|\int_\Omega \rho(s)\overline{g(s)}d\mu(s)\right|^2\right)^{\frac{1}{2}},
\end{aligned} \tag{3.2}$$

and, for the last identity, we also have used Korkine's identity for one function $(f = g)$.

Applying Corollary 1 for the function $g(s) = 1$, $s \in \Omega$ and taking into account that $f(s) \in \bar{D}(\varphi,\delta)$ for $\mu-$a.e. $s \in \Omega$, then we can state that

$$0 \le \int_\Omega \rho(s)|f(s)|^2 d\mu(s) - \left|\int_\Omega \rho(s) f(s)\,d\mu(s)\right|^2 \le \frac{\delta^2}{|\varphi|^2-\delta^2}\left|\int_\Omega \rho(s) f(s)\,d\mu(s)\right|^2. \tag{3.3}$$

Utilising (3.1) and (3.3), we deduce the desired result (3.1).

The fact that $c = 1$ is the best constant is obvious by Corollary 1 and we omit the details. ∎

The following corollary is of interest for applications.

**Corollary 4.** *Assume that $\rho$ is as in Theorem 2. If $f, g \in L^2_\rho(\Omega, \mathbb{K})$ and there exists the constants $\varphi, \Phi \in \mathbb{K}$ with $\operatorname{Re}(\Phi\bar{\varphi}) > 0$ and*

$$\left| f(s) - \frac{\varphi + \Phi}{2} \right| \leq \frac{1}{2} |\Phi - \varphi| \quad \text{for } \mu - a.e.\ s \in \Omega \tag{3.4}$$

*or, equivalently,*

$$\operatorname{Re}\left[ (\Phi - f(s)) \left( \overline{f(s)} - \overline{\varphi} \right) \right] \geq 0 \quad \text{for } \mu - a.e.\ s \in \Omega, \tag{3.5}$$

*then we have the inequality*

$$\left| \int_\Omega \rho(s) f(s) \overline{g(s)} d\mu(s) - \int_\Omega \rho(s) f(s) d\mu(s) \cdot \int_\Omega \rho(s) \overline{g(s)} d\mu(s) \right| \tag{3.6}$$
$$\leq \frac{1}{2} |\Phi - \varphi| \left| \int_\Omega \rho(s) f(s) d\mu(s) \right| \left[ \int_\Omega \rho(s) |g(s)|^2 d\mu(s) - \left| \int_\Omega \rho(s) \overline{g(s)} d\mu(s) \right|^2 \right]^{\frac{1}{2}}.$$

*The constant $\frac{1}{2}$ is best possible in (3.6).*

## 4. Applications for Unidimensional, Finite Fourier Transform

The Fourier transform has long been a principle analytical tool in such diverse fields as linear systems, optics, random process modelling, probability theory, quantum physics and boundary-value problems [1].

In what follows we briefly mention some approximation results for the finite Fourier transform whose proofs have employed recent techniques and facts from Integral Inequalities Theory of Ostrowski type.

Let $g : [a, b] \to \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ be a Lebesgue integrable mapping defined on the finite interval $[a, b]$ and $\mathcal{F}(g)$ its finite Fourier transform, i.e.,

$$\mathcal{F}(g)(t) := \int_a^b g(s) e^{-2\pi i t s} ds.$$

The following inequality was obtained in [2].

**Theorem 3.** *Let $g$ be an absolutely continuous mapping on $[a, b]$. Then we have the inequality*

$$\left| \mathcal{F}(g)(x) - E(-2\pi i x a, -2\pi i x b) \int_a^b g(t) dt \right| \tag{4.1}$$
$$\leq \begin{cases} \frac{1}{3} \|g'\|_\infty (b-a)^2 & \text{if } g' \in L_\infty[a, b]; \\ \frac{2^{\frac{1}{q}}}{[(q+1)(q+2)]^{\frac{1}{q}}} (b-a)^{1+\frac{1}{q}} \|g'\|_p & \text{if } g' \in L_p[a, b], \\ & p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ (b-a) \|g'\|_1, \end{cases}$$

*for all* $x \in [a,b]$, $(x \neq 0)$ *where* $E$ *is the exponential mean of two complex numbers, that is,*

$$E(z,w) := \begin{cases} \dfrac{e^z - e^w}{z-w} & \text{if } z \neq w, \\ \exp(w) & \text{if } z = w, \end{cases} \quad z,w \in \mathbb{C}. \tag{4.2}$$

For functions of bounded variation, the following result holds as well (see [3]):

**Theorem 4.** *Let* $g : [a,b] \to \mathbb{K}$ *be a mapping of bounded variation on* $[a,b]$. *Then we have the inequality*

$$\left| \mathcal{F}(g)(x) - E(-2\pi i x a, -2\pi i x b) \int_a^b g(s)\,ds \right| \leq \frac{3}{4}(b-a) \bigvee_a^b (g), \tag{4.3}$$

*for all* $x \in [a,b]$, $x \neq 0$, *where* $\bigvee_a^b (g)$ *is the total variation of* $g$ *on* $[a,b]$.

Finally, we mention the following result obtained in [5] providing an approximation of the Fourier transform for Lebesgue integrable functions:

**Theorem 5.** *Let* $g : [a,b] \to \mathbb{K}$ *be a measurable function on* $[a,b]$. *Then we have the estimates:*

$$\begin{aligned} &\left| \mathcal{F}(g)(x) - E(-2\pi i x a, -2\pi i x b) \int_a^b g(s)\,ds \right| \\ &\leq \begin{cases} \frac{2\pi}{3} |x| (b-a)^2 \|g\|_\infty & \text{if } g \in L_\infty [a,b]; \\ \frac{2^{1+\frac{1}{q}} (b-a)^{1+\frac{1}{q}}}{[(q+1)(q+2)]^{\frac{1}{q}}} |x| \|g\|_p & \text{if } g \in L_p [a,b], \\ & p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\ 2\pi |x| (b-a) \|g\|_1, & \text{if } g \in L_1 [a,b] \end{cases} \end{aligned} \tag{4.4}$$

*for all* $x \in [a,b]$, $x \neq 0$.

The following result for complex valued functions that illustrate the usefulness of the pre-Grüss type inequality (3.6), holds.

**Theorem 6.** *Let* $g : [a,b] \to \mathbb{K}$ *be a real or complex-valued function with* $g \in L^2([a,b];\mathbb{K})$ *and there exists the constants* $\varphi, \Phi \in \mathbb{K}$ *with the property that* $\operatorname{Re}(\Phi\bar{\varphi}) > 0$ *and, either*

$$\left| g(s) - \frac{\varphi + \Phi}{2} \right| \leq \frac{1}{2} |\Phi - \varphi| \quad \text{for a.e. } s \in [a,b], \tag{4.5}$$

*or, equivalently,*

$$\operatorname{Re}\left[ (\Phi - g(s)) \left( \overline{g(s)} - \bar{\varphi} \right) \right] \geq 0 \quad \text{for a.e. } s \in [a,b], \tag{4.6}$$

*holds. Then we have the inequality:*

$$\left|\mathcal{F}(g)(x) - E(-2\pi ixa, -2\pi ixb)\int_a^b g(s)\,ds\right| \tag{4.7}$$
$$\leq \frac{1}{2}|\Phi-\varphi|\left|\int_a^b g(s)\,ds\right|\left[1-\frac{\sin^2[\pi x(b-a)]}{(b-a)^2\pi^2x^2}\right]^{\frac{1}{2}}(b-a)$$
$$\leq \frac{1}{2}|\Phi-\varphi|\left[1-\frac{\sin^2[\pi x(b-a)]}{(b-a)^2\pi^2x^2}\right]^{\frac{1}{2}}$$
$$\times\begin{cases}(b-a)^2\|g\|_{\infty,[a,b]} & \text{if } g\in L_\infty[a,b];\\ (b-a)^{3/2}\|g\|_{p,[a,b]} & \text{if } g\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ (b-a)\|g\|_{1,[a,b]}, & \text{if } g\in L_1[a,b]\end{cases}$$

*for each $x\in[a,b]$ $(x\neq 0)$, where $E(\cdot,\cdot)$ is the exponential mean defined above.*

*Proof.* We apply the pre-Grüss inequality (3.6) to get:

$$\left|\frac{1}{b-a}\int_a^b g(s)e^{-2\pi ixs}ds - \frac{1}{b-a}\int_a^b e^{-2\pi ixs}ds\cdot\frac{1}{b-a}\int_a^b g(s)\,ds\right| \tag{4.8}$$
$$\leq \frac{1}{2}|\Phi-\varphi|\left|\int_a^b g(s)\,ds\right|\times\left[\frac{1}{b-a}\int_a^b\left|e^{2\pi ixs}\right|^2ds-\left|\frac{1}{b-a}\int_a^b e^{2\pi ixs}ds\right|^2\right].$$

However

$$\int_a^b e^{-2\pi ixs}ds = (b-a)E(-2\pi ixa,-2\pi ixb),$$
$$\left|e^{-2\pi ixs}\right|^2 = 1,$$
$$\int_a^b e^{2\pi ixs}ds = \frac{1}{2\pi ix}\left[e^{2\pi ixb}-e^{2\pi ixa}\right]$$

and

$$\begin{aligned}\left|\int_a^b e^{2\pi ixs}ds\right|^2 &= \frac{1}{(2\pi x)^2}\left[\left|e^{2\pi ixb}\right|^2-2\operatorname{Re}\left[e^{2\pi ixb}\cdot e^{-2\pi ixa}\right]+\left|e^{2\pi ixa}\right|^2\right]\\ &= \frac{1}{4\pi^2x^2}\left[2-2\operatorname{Re}\left[e^{2\pi ix(b-a)}\right]\right]\\ &= \frac{1}{2\pi^2x^2}\left[1-\operatorname{Re}\left[\cos(2\pi x(b-a))+i\sin(2\pi x(b-a))\right]\right]\\ &= \frac{1}{2\pi^2x^2}\left[1-\cos(2\pi x(b-a))\right]\\ &= \frac{1}{2\pi^2x^2}\left[1-\left(1-2\sin^2(\pi x(b-a))\right)\right]\\ &= \frac{\sin^2[\pi x(b-a)]}{\pi^2x^2}.\end{aligned}$$

Using (4.8) multiplied with $b-a>0$, we deduce the first result in (4.7). The second part is obvious by Hölder's inequality and we omit the details. ∎

**Remark 3.** *If $g$ takes real values, then the condition (4.5) may be replaced by the equivalent and more practical condition*

$$\varphi \leq g(s) \leq \Phi \quad \text{for a.e. } s \in [a,b]. \tag{4.9}$$

*provided $\Phi \geq \varphi$.*

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